Indian Statistical Institute, Bangalore Centre
B.Math. (III Year) : 2010-2011
Semester I : Mid-Semestral Examination
Probability III (Stochastic Processes)

4.10.2010 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

Note: The paper carries 82 marks. Any score above 80 will be taken as 80. State clearly the results you are using in your answers.

1. [4+10+7+7 marks] Let $0 < a < 1, 0 < b < 1$. Let $\{X_n\}$ be a Markov chain on $S = \{1, 2\}$ with transition probabilities $P_{11} = 1 - a, P_{12} = a, P_{21} = b, P_{22} = 1 - b$. Let $T_1 = \min\{n \geq 1 : X_n = 1\}$ = first hitting time of state 1.

(i) Show that the Markov chain is irreducible and aperiodic.

(ii) Find $P(T_1 = n|X_0 = 1), n \geq 1$.

(iii) Find $E(T_1|X_0 = 1)$.

(iv) Find the stationary probability distributions, if any.

2. [12 marks] Let $\{X_n : n \geq 0\}$ be a time-homogeneous Markov chain on a countable state space S. Let $y \in S$ be fixed. Put $T_y^{(1)} = T_y = \min\{n \geq 1 : X_n = y\}, T_y^{(2)} = \min\{n > T_y^{(1)} : X_n = y\}$. For any $x \in S, m, n \geq 1$ show that

$$P_x(T_y^{(1)} = m, T_y^{(2)} = m + n) = P_x(T_y = m) \cdot P_y(T_y = n).$$

3. [6+9+6+9 marks] Let $\{X_n\}$ be a Markov chain on a countable state space S with transition probability matrix P. For $x, y \in S$, let $G_n(x, y)$ = expected number of visits to y during times $j = 1, 2, \cdots, n$ for the Markov chain starting at x, and $G(x, y)$ = expected number of visits to y for the Markov chain starting at x..

(i) Show that $G_n(x, y) = \sum_{k=1}^{n} P_{xy}^{(k)}$.

(ii) If y is transient show that $G(x, y) < \infty$ for all $x \in S$.

(iii) If y is transient show that $\lim_{n\to\infty} \frac{1}{n} G_n(x, y) = 0$.

(iv) If π is a stationary probability distribution show that $\pi(x) = \sum_{z\in S} \pi(z) \frac{1}{n} G_n(z, x)$, for any $n \geq 1, x \in S$.

4. [12 marks] Let $\{X_n\}$ be an irreducible Markov chain on a countable state space. Show that $\{X_n\}$ is positive recurrent if and only if it has a unique stationary probability distribution.